Exercise 29-69: A capacitor with resistivity

A capacitor has two parallel plates with area A separated by a distance d. The space between plates is filled with a material having dielectric constant ε_r. The material is not a perfect insulator but has resistivity ρ. THe capacitor is initially charged with charge of magnitude Q_0 on each plate, which gradually discharges by conduction through the dielectric.

1. Calculate the conduction current density $j_c(t)$ in the dielectric.

2. Show that at any instant the displacement current density in the dielectric is equal in magnitude to the conduction current density but opposite in direction, so the *total* current density is zero at every instant.

1 Calculate the conduction density $j_c(t)$ in the dielectric

The conduction current is, according to Ohm's law, equal to the voltage divided by the resistance:

$$i_c = \frac{v}{R} = \frac{v}{\rho\frac{d}{A}} = \frac{vA}{\rho d}$$

The voltage can be obtained by integrating the electrical field E, so first compute that.

$$D = \sigma_{free} = \frac{q}{A}$$

$$E = \frac{D}{\varepsilon_0\varepsilon_r} = \frac{q}{\varepsilon_0\varepsilon_r A}$$

Now integrate to obtain the voltage v.

$$v = -\int_d^0 E dl = -\int_d^0 \frac{q}{\varepsilon_0\varepsilon_r A} dl = \frac{qd}{\varepsilon_0\varepsilon_r A}$$

This is enough to formulate the differential equation with the conduction current. The conduction current i_c is equal to the negative time derivative of the charge, because the current is caused by decreasing charge.

$$i_c = -\frac{dq}{dt} = -\frac{qdA}{\varepsilon_0\varepsilon_r\rho dA} = -\frac{q}{\varepsilon_0\varepsilon_r\rho}$$

Now solve it using the condition that $q = Q_0$ at $t = 0$.

$$q = c_1 e^{-\frac{t}{\varepsilon_0\varepsilon_r\rho}}$$

$$Q_0 = c_1 e^{-\frac{0}{\varepsilon_0\varepsilon_r\rho}} = c_1$$

And we get the final expression for the conduction current density:

$$j_c(t) = \frac{i_c}{A} = \frac{Q_0}{\varepsilon_0\varepsilon_r\rho A} e^{-\frac{t}{\varepsilon_0\varepsilon_r\rho}}$$

2 Compute the displacement current density in the dielectric

The displacement current is given by

$$i_D = \varepsilon_0 \varepsilon_r \frac{d\Phi_E}{dt} = \varepsilon_0 \varepsilon_r A \frac{dE}{dt}$$

We already have computed the electrical field E, but know the charge $q(t)$ is known:

$$E(t) = \frac{q(t)}{\varepsilon_0 \varepsilon_r A} = \frac{Q_0}{\varepsilon_0 \varepsilon_r A} e^{-\frac{t}{\varepsilon_0 \varepsilon_r \rho}}$$

Now we can get an expression for the displacement current i_D

$$i_D = -\varepsilon_0 \varepsilon_r A \frac{Q_0}{\varepsilon_0^2 \varepsilon_r^2 A \rho} e^{-\frac{t}{\varepsilon_0 \varepsilon_r \rho}} = -\frac{Q_0}{\varepsilon_0 \varepsilon_r \rho} e^{-\frac{t}{\varepsilon_0 \varepsilon_r \rho}}$$

The displacement current density j_D is only one more step, dividing by the surface area A:

$$j_D(t) = \frac{i_D}{A} = -\frac{Q_0}{\varepsilon_0 \varepsilon_r \rho A} e^{-\frac{t}{\varepsilon_0 \varepsilon_r \rho}}$$

You see this has the same magnitude as the conduction current density $j_C(t)$ and has the opposite sign.

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